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SPIN L-FUNCTIONS FOR GSO_{10} AND GSO_{12}

BY

J. Hundley

Mathematics Department, Mailcode 4408
Southern Illinois University Carbondale
1245 Lincoln Drive Carbondale, IL 62901
e-mail: jhundley@math.siu.edu

ABSTRACT

Two multi-variable Rankin-Selberg integrals are studied. They may be regarded as extending the theory begun in [G-H1]. Each is shown to be Eulerian with the unramified contribution given explicitly in terms of partial Langlands L-functions.

1. Introduction

In this paper we consider two Rankin–Selberg integrals which were discovered by David Ginzburg, and announced in [G-H1]. These integrals are defined on a split form of GSO_{2n} (see below for precise definition), and involve a generic cuspidal automorphic representation of this group. We content ourselves with showing that both integrals unfold to Eulerian integrals involving Whittaker functions, and computing the contributions from the unramified places. In each case we get a product of two partial Langlands L functions, at least one of which is a “Spin” L-function.

Recall that a Langlands L function requires two pieces of data. The first is an automorphic representation π defined on some group G, from which we obtain a family, indexed by all but finitely many places of our global field, of semisimple conjugacy classes in a certain complex Lie group ${}^LG$. The second is a finite dimensional representation r of that complex Lie group. Recall also that the special orthogonal group SO_{2n} is not simply connected, but possesses a

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simply connected double cover, known as the spin group. This group possesses two fundamental representations, usually called the half-spin representations, which do not factor through the projection. By a Spin L function, we mean a Langlands L function in which the role of r is played by either of these two representations.

In this paper we consider an integral on GSO_{10} and a similar one on GSO_{12}. In both cases we unfold the integral and compute the contribution from the unramified places (being Archimedean is treated as a form of ramification), obtaining a product of partial Langlands L functions. These are: in the GSO_{10} case

$$L^S(3s_1 - 2s_2, \pi, Spin^-)L^S(3s_1 + 2s_2 - 2, \pi, Spin^+)$$

and in the GSO_{12} case

$$L^S(5s_2 - 2, \pi \otimes \chi_2, St)L^S(4s_1 - 3/2, \pi \otimes \chi_1, Spin).$$

The reason for considering nontrivial characters in one case but not the other is explained below.

As the two half-spin representations are related by a symmetry of the Dynkin diagram, what one can prove about one L function follows for the other, and so it is customary in the field to refer to either of these L functions as "the" Spin L function. In this paper we have to be a bit more careful because one of our integrals yields the product of the two half-spin L functions, and while the distinction between one and the other may be safely blurred, the distinction between one of each and two of the same may not.

There are several other known constructions of Spin L functions associated to representations on even orthogonal groups, along the lines of those in this paper. The constructions of Ginzburg in [G] give the same Spin L functions we obtain here by themselves, rather than in a product. In [G-H1] a threefold product was obtained, of the Standard L function and two copies of the same half-spin, each with a different complex-variable argument. A close cousin of this construction was discovered by Wee Teck Gan and studied in [Ga-H]. It is defined on a quasi-split adjoint group of type D_4. In the split case it gives the product of the three L functions associated to the three 8 dimensional representations of $Spin_8(\mathbf{C})$ i.e., the standard and the two half-spins once each. When G is not split, ${}^LG$ is more complicated and its action on this 24 dimensional space has one or two irreducible components. The same construction gives the L function, or product of two associated to this action. Finally, in [G-H2] a construction is given for the L function associated to an automorphic form on the group $GSO_{10}\times PGL_2$, with the representation of ${}^LG$, which in that case is $GSpin_{10}(\mathbf{C})\times SL_2(\mathbf{C})$ being the 32 dimensional tensor product of the Spin representation of $GSpin_{10}(\mathbf{C})$ and the standard representation of $SL_2(\mathbf{C})$.

Please note that, with the exception of the case in [Ga-H] when the 24 dimensional representation of ${}^LG$ is irreducible, all of these L functions have also been studied via the Langlands–Shahidi method [Sh]. In addition, there is also a Spin L function associated to automorphic forms on symplectic groups, which has been studied more extensively. Rather than attempt an independent survey we refer the reader to those of Professor Bump [Bu1, Bu2], in particular Section 13 of [Bu2]. In the theory of automorphic forms on symplectic groups, one encounters a mixture of papers written in "general G" language and papers written in the classical language of Siegel modular forms. The paper of Asgari and Schmidt explains the relationships clearly. Note that if Π (defined on GSO_{2n}) is a weak functorial lift of π (defined on GSp_{2n-2}) associated to the embedding $Spin_{2n-1}(\mathbf{C}) \hookrightarrow Spin_{2n}(\mathbf{C})$ then the two partial Spin L functions of Π agree with one another and with the partial Spin L function of π.

Next we address the question of whether our integrals here might have applications, relating periods, poles of L functions, and functorial liftings, along the lines of [G-R-S], [G-H1] and [G-H2]. As applied to our GSO_{10} integral, this question may be easily answered in the negative: it is proved in [G] that the L functions we obtain in that case are always entire (even without the restriction on central character). For GSO_{12}, on the other hand, what Ginzburg proves is that $L^S(s,\pi\otimes\chi,Spin)$ can have a simple pole when $\omega_\pi\chi^2$ is nontrivial and quadratic. This is the reason why we allow nontrivial characters in one case and not the other: for the GSO_{10} case it is a harmless restriction which simplifies the notation somewhat, while in the GSO_{12} case it omits the most interesting cases from consideration. A possible explanation for this phenomenon arises naturally in connection to the question we consider here. We remark on the structure of the proofs in [G-H1] and [G-H2]. In each case we relate three things:

(1) A partial L function or some partial L functions having poles.
(2) The cuspidal representations that appear in them being lifts associated with the inclusion of the stabilizer of a generic point.
(3) Nonvanishing of a period.

This motivates the investigation of the stabilizer of a generic point in the Spin representation of $GSpin_{12}(\mathbf{C})$. Most of the work is done by Igusa [I] who describes the orbits for the action of $Spin_{12}(\mathbf{C})$ and shows that the stabilizer of a generic point is isomorphic to $SL_6(\mathbf{C})$. One may easily check that in $GSpin$ there is a second connected component; the stabilizer of a generic point is isomorphic to $SL_6(\mathbf{C}) \rtimes \{\pm 1\}$. Also, the stabilizer of a generic point in the standard representation is $GSpin_{11}(\mathbf{C})$ and the intersection of these two groups is $SL_5(\mathbf{C}) \rtimes \{\pm 1\}$. Note that $SL_n(\mathbf{C}) \rtimes \{\pm 1\}$ is essentially the L group of a quasi-split unitary group. On the L-group side we have the diagram of inclusions:

$$(1) \qquad \begin{array}{ccc} SL_5(\mathbf{C}) \rtimes \{\pm 1\} & \xrightarrow{\iota_1} & SL_6(\mathbf{C}) \rtimes \{\pm 1\} \\ {\scriptstyle \iota_2}\big\downarrow & & {\scriptstyle \iota_3}\big\downarrow \\ GSpin_{11}(\mathbf{C}) & \xrightarrow{\iota_4} & GSpin_{12}(\mathbf{C}) \end{array}$$

which indicates which liftings we need to consider. Our integral is best suited to studying the lifting associated with the composite inclusion, but might also be used on the right-hand arrow, the bottom arrow having been handled already in [G-R-S].

In the proofs in [G-H1] and [G-H2], the flow is (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2). The integral representation is a tool for proving the implication (1) $\Rightarrow$ (3). Indeed, obtaining the nonvanishing of a period from the integral representation is immediate, at least if we allow a very vague notion of "period," as seems appropriate. However, one will want to identify a period for which (3) $\Rightarrow$ (2) is true. To do this, one needs some sort of analytic "handle" on the lifting, e.g. by the theta correspondence. One will then want to prove (1) $\Rightarrow$ (3) for this same period. At present the author is unaware of any such handle on the liftings associated with the vertical arrows in (1).

For the implication (2) $\Rightarrow$ (1) we restrict a representation of the L group to a stabilizer. One of the components is the trivial representation corresponding to the stabilized point. We need to know that the L functions attached to the other components of this restriction do not vanish.

Motivated by this, we record the decompositions of the various restrictions. Both semidirect products have a one-dimensional trivial representation which we denote by **1** and a nontrivial one-dimensional representation with kernel equal to the identity component, which we denote by ε. The remaining representations arising here may be described by giving their restrictions to the identity component: there is only one way for -1 to act. We denote the standard representation of $SL_n(\mathbf{C})$ by V_n.

Restricting St to $SL_6(\mathbf{C}) \rtimes \{\pm 1\}$ yields $V_6 \oplus V_6^*$, which is irreducible. When we restrict further to $SL_5(\mathbf{C}) \rtimes \{\pm 1\}$, we get $\mathbf{1} \oplus \varepsilon \oplus (V_5 \oplus V_5^*)$. We insert parentheses because $(V_5 \oplus V_5^*)$ is a single irreducible representation of the semidirect product. Similarly, when $Spin$ is restricted to $SL_6(\mathbf{C}) \rtimes \{\pm 1\}$, we get $\mathbf{1} \oplus \varepsilon \oplus (\bigwedge^2 V_6 \oplus \bigwedge^4 V_6)$, and restricting further to $SL_5(\mathbf{C}) \rtimes \{\pm 1\}$, we get $\mathbf{1} \oplus \varepsilon \oplus (\bigwedge^2 V_5 \oplus \bigwedge^4 V_5) \oplus (V_5 \oplus V_5^*)$.

Now let us describe the notation used in the paper and give the precise statement of the main theorem. We consider the group $G = GSO_{2n}$ generated by matrices preserving the bilinear form given by the matrix J with ones on the diagonal running from upper right to lower left, together with matrices of the form $diag(\lambda I_n, I_n)$. This is a split form of GSO_{2n}. The set of diagonal matrices in this group is a maximal torus, which we denote by T and the set of upper triangular matrices in this group is a Borel subgroup $B = TU$. We define the notation, $e'_{i,j} = e_{i,j} - e_{2n+1-i,2n+1-j}$, where $e_{i,j}$ is the matrix with a one in the i, j entry and zeros elsewhere. For each root α, for the action of T on G, the one dimensional unipotent subgroup on which T acts by α is the image of the homomorphism $x_\alpha(r) = x_{i,j}(r) = I + re'_{i,j}$, for some i, j. We denote this subgroup by X_α or by $X_{i,j}$ as convenient. We number the simple positive roots determined by our choice of Borel $\alpha_1, \ldots, \alpha_n$ so that $X_{\alpha_i} = X_{i,i+1}$ for $i = 1, \ldots, n-1$, and $X_{\alpha_n} = X_{n-1,n+1}$. We identify the Weyl group with the group of permutation matrices that are in G, and for $i = 1, \ldots, n$, let $w[i]$ denote the simple reflection corresponding to the root α_i. We shall write $w[i_1 i_2 \ldots i_r]$ for $w[i_1]w[i_2] \ldots w[i_r]$. Let $M(i_1, \ldots, i_k)$ denote the standard Levi containing the subgroups X_{α_i}, for $i = i_1, \ldots, i_k$, and let $P(i_1, \ldots, i_k)$ denote the standard parabolic of which it is a Levi subgroup.

Let Z denote the center of G. Let $P = P(1, 2, \ldots, n-1)$ and when $n \geq 4$, let $Q = P(1, 2, 4, \ldots, n)$. Let π denote an irreducible cuspidal representation of $G(\mathbf{A})$, and φ a vector in the space of π. We consider two integrals, which correspond to the cases $n = 5$ and $n = 6$. When $n = 5$, we assume that the central character ω_π of π is trivial, while when $n = 6$ we do not. When $n = 5$, we let $E_Q(g, s_1)$ denote the Eisenstein series on $G(\mathbf{A})$ associated to the induced representation $Ind_{Q(\mathbf{A})}^{G(\mathbf{A})}\delta_Q^{s_1}$, and $E_P(g,s_2)$ the one associated to the induced representation $Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\delta_P^{s_2}$. We consider the integral

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)E_Q(g,s_1)E_P(g,s_2)dg. \tag{2}$$

When $n=6$, the integral is much the same except that we allow π to have nontrivial central character, and allow nontrivial characters in the Eisenstein series as well. It has the form

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)E_Q(g,\chi_1')E_P(g,\chi_2')\chi_3'(\lambda(g))dg, \tag{3}$$

where λ is the rational character of G giving the similitude factor, and χ_i' are quasicharacters chosen so that the integrand is $Z(\mathbf{A})$-invariant. See Section 4 for precise notation.

As a final piece of notation, we will need to fix a character of $F\backslash\mathbf{A}$, which we will denote by ψ. We then define a character, also denoted by ψ, of the group U by $\psi(u)=\psi(u_{1,2}+\cdots+u_{n-1,n}+u_{n-1,n+1})$. We let W_φ denote the image of φ in the (U,ψ)-Whittaker model of π. Our integral will be zero unless W_φ is nonzero, so we assume π is generic.

The L-group of GSO_{2n} is $GSpin_{2n}(\mathbf{C})$. For $n=5$ we assume the central character is trivial and hence may work with $Spin_{10}(\mathbf{C})$ instead. We let $Spin^-$ and $Spin^+$ denote the 16 dimensional representations of this group whose highest weights are the fourth and fifth fundamental weights, respectively. When $n=6$, we specify a representation of $GSpin_{12}(\mathbf{C})$ by describing the action of $Spin_{12}(\mathbf{C})$ and the scalars. Specifically, we let St be the 12 dimensional representation where $Spin_{12}$ acts by the standard representation and scalars act trivially. We let $Spin$ denote the representation where $Spin_{12}(\mathbf{C})$ acts by the representation associated to the fifth fundamental weight and scalars act by multiplication.

Our main theorem is then as follows,

Theorem: *When $n=5$, (resp., 6) the integral* (2) *(resp.,* (3)*) unfolds to give an Eulerian integral involving the Whittaker function W_φ. When $n=5$, the contribution from the unramified places is the quotient of*

$$L^S(3s_1-2s_2,\pi,Spin^-)L^S(3s_1+2s_2-2,\pi,Spin^+)$$

by the product of the normalizing factors of the two Eisenstein series, and when $n = 6$ it is the quotient of

$$L^S(5s_2 - 2, \pi \otimes \chi_2, St)L^S(4s_1 - 3/2, \pi \otimes \chi_1, Spin)$$

by the product of the normalizing factors of the two Eisenstein series.

Here the term **normalizing factor** is used as follows: the poles of an Eisenstein series are determined by the constant term, which is given in terms of intertwining operators which factor over the places. At an unramified place this intertwining operator takes the normalized spherical vector to a multiple of the normalized spherical vector, with the multiplier being given by a ratio of products of local zeta functions. Taking the product over all the unramified places we obtain a ratio of products of partial zeta functions. The normalizing factor is the product appearing in the denominator.

We now describe the format of the paper. Sections 2 and 3 are devoted to the case $n = 5$, with Section 2 being the unfolding and Section 3 being the unramified computation. Similarly, Sections 4 and 5 are the unfolding and unramified computation in the case $n = 6$ respectively.

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2. The GSO_{10} Integral

The unfolding in this case was communicated by Ginzburg. Any mistakes are the author's own. For $g \in G(\mathbf{A})$, $f_{Q,s_1} \in Ind_{Q(\mathbf{A})}^{G(\mathbf{A})}\delta_Q^{s_1}$ and $f_{P,s_3} \in Ind_{P(\mathbf{A})}^{G(\mathbf{A})}\delta_P^{s_2}$ we define

(4)

$$f^R_{Q,s_1}(g) := \int_{A^4} f_{Q,s_1}(w[3254]x_{23}(r_1)x_{45}(r_2)x_{46}(r_3)x_{27}(r_4)g)\psi^{-1}(r_1+r_2+r_3)dr_i$$

and

$$(5)\quad f^L_{P,s_2}(g) := \int_{A^6} f_{P,s_2}(w[532143]x_{12}(l_1)x_{34}(l_2)x_{14}(l_3), x_{15}(l_4)x_{35}(l_5)x_{18}(l_6)g)\psi^{-1}(l_1+l_2)dl_i$$

The main result of this section is the following

THEOREM 2.1: *For $Re(s_i)$ large, we have*

$$(6)\quad \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)E_Q(g,s_1)E_P(g,s_2)dg = \int_{Z(\mathbf{A})U(\mathbf{A})\backslash G(\mathbf{A})} W_\varphi(g)f^R_{Q,s_1}(g)f^L_{P,s_2}(g)dg$$

Proof. We unfold the two Eisenstein series and obtain

$$\sum_{w\in Q\backslash G/P} \int_{(P(F)\cap w^{-1}Q(F)w)\backslash G(\mathbf{A})} \varphi(g)f_{s_1}(wg)f_{s_2}(g)dg.$$

By the Bruhat decomposition, every double coset in $Q\backslash G/P$ contains an element of the Weyl group of G. The Weyl group of G may be identified with the set of permutations with sign 1 such that $w(11-i) = 11 - w(i)$ for all i. From the block structure of P and Q we see that there are four elements of $Q\backslash G/P$ corresponding to the four possible values of $\#\{i : i \le 5, w(i) \le 3\}$. For each double coset we choose the shortest element of the Weyl group in that coset as a representative. Then the unipotent radical of $P(2,3,4,5)$ is contained in $(P(F)\cap w^{-1}Q(F)w)$ for every coset but one. By cuspidality, all those integrals vanish. Our representative for the remaining coset is $w_0 = w[321532435]$. The group $(P(F)\cap w_0^{-1}Q(F)w_0)$ consists of $M(1,3,4)$ and the 7 dimensional unipotent group containing X_{ij} for $i = 1,2, j = 3,4,5$, as well as $(1,9)$. We make the change of variables $g \mapsto w[534]g$. The effect on the domain of integration is to conjugate the "denominator" $(P(F)\cap w_0^{-1}Q(F)w_0)$ by $w[435]$. The group $M(1,3,4)$ maps to $M(1,3,5)$ and the unipotent subgroup now contains X_{ij} for $i = 1,2, j = 5,7,8,9$.

Next we perform a Fourier expansion of φ along the three dimensional unipotent subgroup $X_{35}X_{45}X_{47}$. Together with the unipotent subgroup we already have, this forms $U(1,2,3,5)$, hence the term corresponding to the trivial character vanishes. The action of $M(1,3,5)$ by conjugation permutes the remaining terms transitively. We choose as a representative $\psi_1(u) = \psi(u_{45})$, which may also be viewed as a character of $U(1,2,3,5)$. The stabilizer of ψ_1 contains the two dimensional unipotent group $X_{34}X_{36}$ and a subgroup M_1 of $M(1,5)$ isomorphic to $GL_1 \times GL_2 \times GL_2$. Thus, (2) is equal to

$$\int_{Z(\mathbf{A})N_1(\mathbf{A})M_1(F)\backslash G(\mathbf{A})} f_{s_1}(w_2 g) f_{s_2}(w_1 g) \varphi^{U_1,\psi_1}(g)dg, \tag{7}$$

where $U_1 = U(1,2,3,5)X_{34}X_{36}$, and N_1 is the product of $X_{34}X_{36}$ and the seven dimensional unipotent group above.

Next, we make the change of variables $g \mapsto w[21]g$. When N_1 is conjugated by $w[12]$, $X_{34}X_{36}$ is sent to to $X_{14}X_{16}$. We expand along $X_{12}X_{13}$. The nontrivial characters are permuted transitively by the action of M_1 on this group, while the trivial character contributes zero by cuspidality. We take the character $x_{12}(r_1)x_{13}(r_2) \mapsto \psi(r_1)$ as a representative. The stabilizer contains X_{23}, and a reductive part M_2 isomorphic to $GL_1^2 \times GL_2$. We then expand along $X_{24}X_{26}$, choosing this time $x_{24}(r_1)x_{26}(r_2) \mapsto \psi(r_1)$ as our representative for the nontrivial orbit. The stabilizer contains X_{46}. Factoring the integration over $X_{23}X_{46}$, we have shown that (7) is equal to

$$\int_{Z(\mathbf{A})N_3(\mathbf{A})M_3(F)\backslash G(\mathbf{A})} f_{s_1}(w_2 g) f_{s_2}(w_1 g) \varphi^{U_3,\psi_3}(g)dg, \tag{8}$$

where $N_3 = X_{23}X_{46}X_{14}X_{16}w[12]N_1w[21]$, U_3 is the unipotent subgroup containing all positive root spaces except X_{34} and X_{36}, $\psi_3(u) = \psi(u_{12}+u_{24}+u_{45})$, and $M_3 \cong GL_1^3$ is the stabilizer of ψ_3 in T.

We change variables $g \mapsto w[34]g$. The group $w[43]U_3w[34]$ consists of X_{54} and the group V_4 which is the product of every positive root space except $(4,5), (4,6)$ and $(3,5)$. If $\psi_4(u)\psi_3(w[34]uw[43])$ for $u = vx_{54}(r) \in w[43]U_3w[34]$ $\psi_4(v) = \psi_4(u) = \psi(u_{12}+u_{23}+u_{34})$. Clearly, $\varphi^{U_3,\psi_3}(w[34]g) = \varphi^{w[43]U_3w[34],\psi_4}(g)$. We express this as an integral over X_{54} and one over the group V_4. generated by all the other root spaces. Then, we expand φ along X_{35}:

$$\sum_{\xi\in F}\int_{(F\backslash\mathbf{A})^2}\int_{V_4(F)\backslash V_4(\mathbf{A})} \varphi(x_{35}(r)vx_{54}(r')g)\psi_4(v)\psi(\alpha r)dvdrdr'.$$

As φ is left $G(F)$-invariant, we may introduce $x_{54}(\alpha)$ at the far left. Now $x_{54}(\alpha)x_{35}(r) = x_{35}(r)x_{34}(\alpha r)x_{54}(\alpha)$. We conjugate x_{54} to the right, and after suitable changes of variable, we obtain

$$\int_{\mathbf{A}} \varphi^{U_4,\psi_4}(x_{54}(r')g)dr'.$$

Where $U_4 = X_{35}V_4$ and we extend the character ψ_4 trivially. The root space X_{54} is in $w[43]N_3[w34]$, so we may collapse the integration. We now have

$$(9) \qquad \int_{Z(\mathbf{A})N_5(\mathbf{A})M_4(F)\backslash G(\mathbf{A})} f_{s_1}(w[3254]g)f_{s_2}(w[5342134]g)\varphi^{U_4,\psi_4}(g)dg,$$

where N_4 is obtained by deleting the root space X_{54} from $w[43]N_3w[34]$, U_5 is the product of all the positive root spaces except X_{45} and X_{46}, and M_4 is the stabilizer of ψ_4 in T. We observe that $w[5342134] = w[2532143]$. The leading two can be deleted as f_{s_2} is $P(F)$- invariant. Finally, we expand along X_{45} and X_{46}, and then factor the unipotent integration $N_4\backslash U$, to obtain the right-hand side of (6). ∎

3. The Unramified Computation for GSO_{10}

We now consider the local unramified integral which results from (9). In this section F will denote a non-archimedean local field, π an unramified irreducible representation of $G(F)$, with trivial central character, and f^R_{Q,s_1} and f^L_{P,s_2} will denote the local analogues of the global functionals defined above. Also, in this section we work exclusively with F points of our various algebraic groups $(G, T, Z,$ etc.) and hence may suppress the "(F)" from the notation.

The integral we consider is

$$(10) \qquad \int_{ZU\backslash G} W_\pi(g)f^R_{Q,s_1}(g)f^L_{P,s_2}(g)dg.$$

The main result of this section is

Proposition: *For all unramified data and $Re(s_i)$ sufficiently large, the integral* (10) *is equal to*

$$\frac{L(3s_1 - 2s_2, \pi, Spin^-)L(3s_1 + 2s_2 - 2, \pi, Spin^+)}{\zeta(6s_1)\zeta(6s_1 - 1)^2\zeta(6s_1 - 2)\zeta(12s_1 - 4)\zeta(8s_2)\zeta(8s_2 - 2)}.$$

The denominator here matches the product of the normalizing factors of the two Eisenstein series exactly.

Proof. By the Iwasawa decomposition, (10) equals

$$(11) \qquad \int_{Z\backslash T} W_\pi(t)f^R_{Q,s_1}(t)f^L_{P,s_2}(t)\delta_B^{-1}(t)dt.$$

We now compute $f^R_{Q,s_1}(t)$, which is defined by

(12)

$$f^R_{Q,s_1}(t) := \int_{F^4} f_{Q,s_1}(w[3254]x_{23}(r_1)x_{45}(r_2)x_{46}(r_3)x_{27}(r_4)t)\psi^{-1}(r_1+r_2+r_3)dr_i.$$

The integration in r_4 gives an intertwining operator from $Ind_Q^G\delta_Q^{s_1}$ to $Ind_B^G\chi_{s_1}$ where

$$\chi_{s_1}(diag(t_1,t_2,t_3,t_4,t_5,t_6,\dots)) = |t_1^2t_2^2t_4^2t_5^{-3}t_6^{-3}|^{3s_1}\Big|\frac{t_3}{t_4}\Big|.$$

Let $f^\circ_{\chi_{s_1}}$ denote the normalized spherical vector in this latter space. Then by a well-known calculation we get

$$(13)\quad f^R_{Q,s_1}(t) := \frac{\zeta(6s_1-1)}{\zeta(6s_1)}\int_{F^3} f^\circ_{\chi_{s_1}}(w[254]x_{23}(r_1)x_{45}(r_2)x_{46}(r_3)t)\psi^{-1}(r_1+r_2+r_3)dr_i.$$

Each of the roots $\alpha_2,\alpha_4,\alpha_5$ defines an embedding of SL_2 into G. As no two of these roots are connected, the images of SL_2 commute. Put differently, we obtain an embedding of SL_2^3 into G. The integration in the remaining three variables essentially gives Whittaker functionals on our three SL_2's.

More explicitly, beginning from (13) we conjugate t to the right, and make a change of variables in the r_i. We now need to evaluate the integral

$$\int_{F^3} f^\circ_{\chi_{s_1}}(w[245]x_{23}(r_1)x_{45}(r_2)x_{46}(r_3))\psi^{-1}\Big(\frac{t_2}{t_3}r_1+\frac{t_4}{t_5}r_2+\frac{t_4}{t_6}r_3\Big)dr_i.$$

We split the integration in r_3 into an integral over the ring of integers $\mathfrak{o}$ and one over $F-\mathfrak{o}$. The first contributes

$$\int_{F^2} f^\circ_{\chi_{s_1}}(w[24]x_{23}(r_1)x_{45}(r_2))\psi^{-1}\Big(\frac{t_2}{t_3}r_1+\frac{t_4}{t_5}r_2\Big)dr_1dr_2\int_{\mathfrak{o}}\psi^{-1}\Big(\frac{t_4}{t_6}r_3\Big)dr_3,$$

while the second gives

$$\int_{F^2}\int_{F-\mathfrak{o}} f^\circ_{\chi_{s_1}}\Big(w[24]x_{23}(r_1)x_{45}(r_2)x_{46}(r_3^{-1})\check{\alpha_5}(r_3^{-1})\Big)\psi^{-1}(\frac{t_2}{t_3}r_1+\frac{t_4}{t_5}r_2+\frac{t_4}{t_6}r_3)dr_i,$$

where $\check{\alpha_5}(r_3^{-1}) = diag(1,1,1,r_3^{-1},r_3^{-1},r_3,r_3,1,1,1)$. This, in turn, is equal to

$$\int_{F^2} f^\circ_{\chi_{s_1}}(w[24]x_{23}(r_1)x_{45}(r_2))\psi^{-1}\Big(\frac{t_2}{t_3}r_1+\frac{t_4}{t_5}r_2\Big)dr_1dr_2 \times\int_{F-\mathfrak{o}}|r_3|^{6s_1-1}\psi^{-1}\Big(\frac{t_5}{t_7}r_3\Big)dr_3.$$

The other variables behave similarly. Since

$$\int_{\mathfrak{o}} \psi^{-1}(\tau r)dr + \int_{F-\mathfrak{o}} |r|^{6s_1-1}\psi^{-1}(\tau r)d = \frac{\zeta(6s_1-2)}{\zeta(6s_1-1)}\Big(1-|\tau|^{6s_1-2}q^{-6s_1+2}\Big),$$

overall we get

$$\frac{\zeta(6s_1-2)^3}{\zeta(6s_1)\zeta(6s_1-1)^2}\left(1-\left|\frac{t_2}{t_3}\right|^{6s_1-2}q^{-6s_1+2}\right)\left(1-\left|\frac{t_4}{t_5}\right|^{6s_1-2}q^{-6s_1+2}\right)$$
$$\times\left(1-\left|\frac{t_4}{t_6}\right|^{6s_1-2}q^{-6s_1+2}\right)|t_1^6t_3^6t_4^{-6}t_5^{-3}t_6^{-3}|^{s_1}|t_2^2t_3^{-1}t_4^3t_5^{-2}t_6^{-2}|.$$

The computation of f^L_{P,s_2} is similar. In this case, the integrals in l_3 to l_6 give intertwining operators, and the integrals in l_1 and l_2 give Whittaker functionals on embedded SL_2's. The outcome is

$$f^L_{P,s_2}(t) = \frac{\zeta(8s_2-4)^2}{\zeta(8s_2)\zeta(8s_2-2)}\left(1-\left|\frac{t_1}{t_2}\right|^{8s_2-4}q^{-8s_2+4}\right)$$
$$\times\left(1-\left|\frac{t_3}{t_4}\right|^{8s_2-4}q^{-8s_2+4}\right)|t_1^{-4}t_2^4t_3^{-4}t_4t_5^2t_6^{-2}|^{s_2}|t_1^4t_2^{-1}t_3^3t_4^{-2}t_5^{-3}t_6^{-1}|.$$

Also

$$\delta_B(t) = \frac{t_1^8t_2^6t_3^4t_4^2}{t_5^{10}t_6^{10}}.$$

Let $\tau_i = t_i/t_{i+1}, i=1$ to 4 and $\tau_5 = t_4/t_6$. Then the variables τ_i define coordinates on $Z\backslash T$. Let $K_\pi(t) = W_\pi(t)\delta_B(t)^{-1/2}$. Then we have shown that (11) is equal to

$$\frac{\zeta(6s_1-2)^3\zeta(8s_2-4)^2}{\zeta(6s_1)\zeta(6s_1-1)^2\zeta(8s_2)\zeta(8s_2-2)}\int_{Z\backslash T} K_\pi(t)\eta(t)dt, \tag{14}$$

where

$$\eta(t) = \prod_{i=2,4,5}(1-|\tau_i p|^{6s_1-2})$$
$$\times\prod_{i=1,3}(1-|\tau_i p|^{8s_2-4})|\tau_1|^{6s_1-4s_2}|\tau_2|^{6s_1-2}|\tau_3|^{12s_1-4s_2-2}|\tau_4|^{3s_1-2s_2}|\tau_5|^{3s_1+2s_2-2}.$$

Next, we use the Casselman-Shalika formula. Let n_i denote the valuation of τ_i, so that $|\tau_i| = q^{-n_i}$. Then the Casselman-Shalika formula states first that $K_\pi(t)$ is zero if any of the n_i is negative, and that if they are all positive, then it is equal to the trace of the irreducible representation of $Spin_{10}(\mathbf{C})$ with highest weight $\sum_i n_i\varpi_i$, where ϖ_i is the ith fundamental weight, evaluated at the semisimple conjugacy class associated to the local unramified representation

π. Finally, we let $x = q^{-3s_1+1}, y = q^{-2s_2+1}$. Putting all of this together, we find that the integral in (14) is equal to

$$\sum_{n_i=0}^{\infty}(n_1,n_2,n_3,n_4,n_5)\prod_{i=2,4,5}(1-x^{2(n_i+1)}) \\ \times\prod_{i=1,3}(1-y^{4(n_i+1)})x^{2n_1+2n_2+4n_3+n_4+n_5}y^{-2n_1-2n_3-n_4+n_5}.$$

It follows from the result of Brion (see [Br] and also [G] p. 781) that

$$L(3s_1+2s_2-2,\pi,Spin^+) = \sum_{m_1,m_5=0}^{\infty}(m_1,0,0,0,m_5)(xy)^{2m_1+m_5}$$

$$L(3s_1-2s_2,\pi,Spin^-) = \sum_{k_1,k_4=0}^{\infty}(k_1,0,0,k_4,0)(xy^{-1})^{2k_1+k_4}.$$

The proposition is now reduced to the identity

(15)
$$\sum_{n_i=0}^{\infty}(n_1,n_2,n_3,n_4,n_5)\prod_{i=2,4,5}\frac{1-x^{2(n_i+1)}}{1-x^2}\prod_{i=1,3}\frac{1-y^{4(n_i+1)}}{1-y^4}y^{-2n_1-2n_3-n_4+n_5} \\ =(1-x^2)(1-x^4) \\ \times\sum_{m_i,k_i=0}^{\infty}(m_1,0,0,0,m_5)(k_1,0,0,k_4,0)x^{2m_1+m_5+2k_1+k_4}y^{2m_1+m_5-2k_1-k_4}.$$

The method of proof is as in [G-H2]. Let

$$P(u) = (1-u^8)+(u^6-u^2)(1,0,0,0,0)+u^3(0,0,0,1,0)-u^5(0,0,0,0,1),$$
$$P'(u) = (1-u^8)+(u^6-u^2)(1,0,0,0,0)+u^3(0,0,0,0,1)-u^5(0,0,0,1,0).$$

Then

$$P(xy)L(3s_1+2s_2-2,\pi,Spin^+) = \sum_{m=0}^{\infty}(0,0,0,0,m)(xy)^m$$

$$P'(xy^{-1})L(3s_1-2s_2,\pi,Spin^-) = \sum_{k=0}^{\infty}(0,0,0,k,0)(xy^{-1})^k.$$

We multiply the left hand side of (15) by $P'(xy^{-1})$. ∎

Lemma: *The outcome is*

$$(16)\quad \sum_{n_i=0}^{\infty}(n_1,n_2,n_3,n_4,n_5)\frac{1-x^{2(n_5+1)}}{1-x^2}x^{2n_1+2n_2+4n_3+n_4+n_5}y^{2n_1+2n_3-n_4+n_5}.$$

Proof of Lemma: We denote the weight $\sum_i n_i\varpi_i$ by $\underline{n}$. Let

$$\ell_1(\underline{n}) = 2n_1 + 2n_2 + 4n_3 + n_4 + n_5,$$
$$\ell_2(\underline{n}) = 2n_1 + 2n_3 - n_4 + n_5.$$

Let

$$h_{\underline{n}}(x,y) = x^{\ell_1(\underline{n})}y^{\ell_2(\underline{n})}\prod_{i=1,3}(1-y^{-4(n_i+1)})\prod_{i=2,4,5}(1-x^{2(n_i+1)}),$$

which, up to a factor of $(1-x^2)^3(1-y^{-4})^2$, is the coefficient of (n_1,n_2,n_3,n_4,n_5) in (15), for $n_i \geq 0$. Also, if any of the n_i is -1, then $h_{\underline{n}}(x,y) = 0$. It follows that the coefficient of (n_1,n_2,n_3,n_4,n_5) in

$$P'(xy^{-1})\sum_{n_i=0}^{\infty}(n_1,n_2,n_3,n_4,n_5)h_{\underline{n}}(x,y)$$

is

$$(1-x^8y^{-8})h_{\underline{n}}(x,y) + (x^6y^{-6}-x^2y^{-2})\sum_{\underline{w}\in\Gamma_1}h_{\underline{n}-\underline{w}}(x,y) + x^3y^{-3}\sum_{\underline{w}\in\Gamma_5}h_{\underline{n}-\underline{w}}(x,y) - x^5y^{-5}\sum_{\underline{w}\in\Gamma_4}h_{\underline{n}-\underline{w}}(x,y),$$

where Γ_i denotes the set of weights of the representation with highest weight ϖ_i. Let

$$H_{\underline{w}} = x^{-\ell_1(\underline{w})}y^{-\ell_2(\underline{w})}(1-Y_1y^{4w_1-4})(1-X_2x^{2-2w_2})(1-Y_3y^{4w_3-4}) \times (1-X_4x^{2-2w_4})(1-X_5x^{2-2w_5}).$$

Then the lemma is equivalent to the identity

$$(1-x^8y^{-8})H_{\underline{0}} + (x^6y^{-6}-x^2y^{-2})\sum_{\underline{w}\in\Gamma_1}H_{\underline{w}} + x^3y^{-3}\sum_{\underline{w}\in\Gamma_5}H_{\underline{w}} - x^5y^{-5}\sum_{\underline{w}\in\Gamma_4}H_{\underline{w}} = (1-x^2)^2(1-y^{-4})^2(1-x^4)(1-X_5x^2).$$

This is just an identity of polynomials (with y^{-1} being one of the variables), and may be verified by the computer algebra system of your choice. ∎

Now we multiply (16) by $P(xy)$.

LEMMA:

(17)
$$P(xy)\sum_{n_i=0}^{\infty}(n_1,n_2,n_3,n_4,n_5)\frac{1-x^{2(n_5+1)}}{1-x^2}x^{2n_1+2n_2+4n_3+n_4+n_5}y^{2n_1+2n_3-n_4+n_5}$$
$$\times\sum_{n_2,n_4,n_5=0}^{\infty}(0,n_2,0,n_4,n_5)x^{2n_2+n_4+n_5}y^{-n_4+n_5}.$$

Proof of Lemma: Let $h'_{\underline{n}}(x,y)=x^{\ell_1(\underline{n})}y^{\ell_2(\underline{n})}(1-x^{2(n_5+1)})$.

The coefficient of (n_1,n_2,n_3,n_4,n_5) on the left hand side of (17) is
$$(1-x^8y^8)h'_{\underline{n}}+(x^6y^6-x^2y^2)\sum_{\underline{w}\in\Gamma_1^{\underline{n}}}h'_{\underline{n}-\underline{w}}+x^3y^3\sum_{\underline{w}\in\Gamma_4^{\underline{n}}}h'_{\underline{n}-\underline{w}}-x^5y^5\sum_{\underline{w}\in\Gamma_5^{\underline{n}}}h'_{\underline{n}-\underline{w}},$$
where
$$\Gamma_i^{\underline{n}}=\{\underline{w}\in\Gamma_i:w_i\le n_i,i\le 4\}.$$
(Note that $|w_i|\le 1$ for all i, and all $\underline{w}$ under consideration. It is not necessary to exclude the terms with $w_5>n_5$ from our sum, because these terms vanish anyway.) We must show that this sum is 0 if n_1 or n_3 is nonzero, and $(1-x^2)x^{\ell_1(\underline{n})}y^{\ell_2(\underline{n})}$ otherwise.

Let
$$H'_{\underline{w}}(x,y,X_5)=x^{-\ell_1(\underline{w})}y^{-\ell_2(\underline{w})}(1-X_5x^{2-2w_5}),$$
so that
$$h'_{\underline{n}-\underline{w}}(x,y)=x^{\ell_1(\underline{n})}y^{\ell_2(\underline{n})}H'_{\underline{w}}(x,y,x^{2n_5}).$$
For each $\sigma\in\{0,1\}^4$ we define
$$\Gamma_i^{\sigma}=\{\underline{w}\in\Gamma_i:\sigma_i=1\Leftrightarrow w_i=1\}.$$
Let
$$Q^{\sigma}=(1-x^8y^8)H'_{\underline{0}}+(x^6y^6-x^2y^2)\sum_{\underline{w}\in\Gamma_1^{\sigma}}H'_{\underline{w}}+x^3y^3\sum_{\underline{w}\in\Gamma_4^{\sigma}}H'_{\underline{w}}-x^5y^5\sum_{\underline{w}\in\Gamma_5^{\sigma}}H'_{\underline{w}}.$$
Then lengthy but straightforward computation shows that
$$Q^{\sigma}(x,y,X_5)=\begin{cases}1-x^2 & \sigma=(0,0,0,0),(1,0,1,0)\\ -1+x^2 & \sigma=(1,0,0,0),(0,0,1,0)\\ 0 & \text{otherwise.}\end{cases}$$
The result follows.

Equation (15) is now reduced to

$$(18)\quad (1-x^2)^{-1}\sum_{n_2,n_4,n_5=0}^{\infty}(0,n_2,0,n_4,n_5)x^{2n_2+n_4+n_5}y^{-n_4+n_5}$$
$$=\sum_{m_5,k_4=0}^{\infty}(0,0,0,k_4,0)(0,0,0,0,m_5)x^{k_4+m_5}y^{-k_4+m_5}.$$

This, in turn, follows from the identity

$$(0,0,0,k_4,0)(0,0,0,0,m_5)=\sum_{\substack{a,b,c:a+\min(b,c)\le\min(k_4,m_5)\\ b-c=k_4-m_5}}(0,a,0,b,c).$$

which is due to Okada [O]. See also [K]. ∎

4. The Global Integral for GSO_{12}

In this integral, we will allow nontrivial characters. Observe that now the Satake parameters may not be in $Spin_{12}(\mathbf{C})\subset GSpin_{12}(\mathbf{C})$.

We define a rational character d_3 of M_Q by

$$d_3\begin{pmatrix} g & & \\ & h & \\ & & * \end{pmatrix}=\det g.$$

Here $*$ is defined by the condition that this matrix is in GSO_{12}. Let P denote the Siegel parabolic. We define a rational character d_6 of M_P by

$$d_6\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix}=\det g_1.$$

Then the lattice of rational characters of Q (resp., P) is generated by d_3 (resp., d_6) and the similitude factor λ, which is a rational character of GSO_{12}. We fix two characters χ_1 and χ_2 of $F\backslash\mathbf{A}$, and let s_1 and s_2 be complex variables. We define three quasicharacter- valued variables depending on this data:

$$\begin{aligned}\chi_1'(r)&=|r|^{8s_1}\chi_1^2(r)\omega_\pi(r)\\ \chi_2'(r)&=|r|^{5s_2}\chi_2(r)\\ \chi_3'(r)&=|r|^{-12s_1-15s_2}\chi_1^{-3}(r)\chi_2^{-3}(r)\omega_\pi^{-2}(r).\end{aligned}$$

Then we consider two Eisenstein series:

$$E_Q(g,\chi_1'),\ \text{associated with } Ind_{Q(\mathbf{A})}^{G(\mathbf{A})}(\chi_1'\circ d_3),$$

and

$$E_P(g,\chi_2'),\ \text{associated with } Ind_{P(\mathbf{A})}^{G(\mathbf{A})}(\chi_2'\circ d_6).$$

We let $f_{\chi'_1}$ and $f_{\chi'_2}$ denote the vectors in these induced spaces, respectively. The integral we consider is

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)E_Q(g,\chi'_1)E_P(g,\chi'_2)\chi'_3(\lambda(g))dg. \tag{19}$$

(Observe that the integrand is indeed $Z(\mathbf{A})$- invariant.)

Let

$$\begin{aligned} f^R_{\chi'_1}(g) = \\ \int_{\mathbf{A}^7} f_{\chi'_1}(w[3423156]x_{12}(r_1)x_{14}(r_2)x_{18}(r_3)x_{34}(r_4)x_{38}(r_5)x_{56}(r_6)x_{57}(r_7)g) \\ \times \psi(r_1+r_4+r_6+r_7)dr_i, \end{aligned} \tag{20}$$

and let

$$\begin{aligned} f^L_{\chi'_2}(g) = \\ \int_{\mathbf{A}^8} f_{\chi'_2}(w[643524]x_{21}(l_1)x_{13}(l_2)x_{23}(l_3)x_{25}(l_4)x_{26}(l_5)x_{29}(l_6)x_{45}(l_7)x_{46}(l_8)g) \\ \times \psi(l_3+l_7)dl_i. \end{aligned}$$

Then we prove

PROPOSITION: *The integral* (19) *is equal to*

$$\int_{Z(\mathbf{A})U(\mathbf{A})\backslash G(\mathbf{A})} W_\varphi(g)f^R_{\chi'_1}(g)f^L_{\chi'_2}(g)\chi'_3(\lambda(g))dg. \tag{21}$$

Proof. We unfold the two Eisenstein series, and analyze the contributions from the double cosets $Q\backslash G/P$. As before, all but one of them contribute zero to the integral. For the one that does not, we choose as a representative the element $w_0 = w[346234512346]$. We obtain,

$$\int_{Z(\mathbf{A})M_0(F)N_0(F)\backslash G(\mathbf{A})} \varphi(g)f_{\chi'_1}(w_0g)f_{\chi'_2}(g)\chi'_3(\lambda(g))dg, \tag{22}$$

where $M_0 = M(1,2,4,5)$ and

$$N_0 = \left\{ \begin{pmatrix} I & C_1 & & C_2 \\ & I & & \\ & & I & C_1^* \\ & & & I \end{pmatrix} : C_1, C_2 \in Mat_{3\times 3} \right\},$$

and C_1^* is defined by the condition that this matrix is in G, which also puts conditions on C_2.

Next, we conjugate by $w_1 = w[546]$. This takes M_0 to $M(1,2,4,6)$, and N_0 to the product of the groups X_{ij}, where $i \le 3$ and $j = 6$ or $j \ge 8$.

Next, we expand φ along $X_{46}X_{48}X_{56}$. The term corresponding to the trivial character is the constant term of φ along $P(1,2,3,4,6)$, hence it contributes zero. The group $M(4,6)$ permutes the remaining characters transitively. We choose as a representative the character $\psi(u_{56})$. Its stabilizer contains $X_{45}X_{47}$. We factor the integration over this group, obtaining

$$(23) \qquad \int_{Z(\mathbf{A})M_1(F)N_1(F)\backslash G(\mathbf{A})} \varphi(g)^{U_1,\psi_{U_1}} f_{\chi_1'}(w_0 w_1^{-1} g) f_{\chi_2'}(w_1^{-1} g)\chi_3'(\lambda(g))dg,$$

where $N_2 = X_{45}X_{47}w_1N_0w_1^{-1}$, $U_1 = X_{45}X_{47}U_{P(1,2,3,4,6)}$, the character ψ_{U_1} is given by $\psi_{U_1}(u) = \psi(u_{56})$, and $M_1 \cong GL_3 \times GL_1 \times GL_2$ is a subgroup of $M(1,2,6)$ given by a relation between the similitude factor and the determinant of the GL_2 component.

Next, we conjugate by $w_2 = w[123]$. This takes $M(1,2,6)$ to $M(2,3,6)$, and U_1 to $X_{1,5}X_{1,7}U_{P(1,2,3,4,6)}$. We then expand φ along $X_{12}X_{13}X_{14}$. The constant term contains integration corresponding to the constant term of φ along $P(2,3,4,5,6)$, and the remaining terms are permuted transitively by $w_2M_1w_2^{-1}$. We choose as a representative $\psi(u) = \psi(u_{12})$. The stabilizer contains $X_{23}X_{24}$. We factor the integration over this group, and obtain

$$(24) \qquad \int_{Z(\mathbf{A})M_2(F)N_2(F)\backslash G(\mathbf{A})} \varphi(g)^{U_2,\psi_{U_2}} f_{\chi_1'}(w_0 w_1^{-1} w_2^{-1} g) f_{\chi_2'}(w_1^{-1} w_2^{-1} g)\chi_3'(\lambda(g))dg,$$

where $U_2 = X_{23}X_{24}U_{P(2,3,4,5)}$, $\psi_{U_2}(u) = \psi(u_{12} + u_{56})$, M_2 is a subgroup of $M(3,6)$ isomorphic to $GL_1 \times GL_2 \times GL_2$, and N_2 is the product of the groups X_{ij} for the following pairs $(i,j) : (1,5), (1,7), (1,9), (1,10), (1,11), (2,3), (2,4)$, and $2 \le i \le 4, j = 6$ or $8 \le j \le 12 - i$.

Next, we expand φ along $X_{25}X_{27}$. The constant term contributes zero by cuspidality. The other characters are permuted transitively by the copy of GL_2 containing X_{57}, and the stabilizer of the character $\psi(u) = \psi(u_{25})$ contains X_{57}. We factor the integration over this group, and obtain

$$(25) \qquad \int_{Z(\mathbf{A})M_3(F)N_3(F)\backslash G(\mathbf{A})} \varphi(g)^{U_3,\psi_{U_3}} f_{\chi_1'}(w_0 w_1^{-1} w_2^{-1} g) f_{\chi_2'}(w_1^{-1} w_2^{-1} g)\chi_3'(\lambda(g))dg,$$

where $U_3 = X_{57}U_{P(3,4,6)}, \psi_{U_3}(u) = \psi(u_{12} + u_{25} + u_{56})$, M_3 is a subgroup of $M(3)$ isomorphic to $GL_2 \times GL_1$, and $N_3 = X_{5,7}N_2$.

The next step is to prove the identity

$$(26)\quad \varphi^{U_3,\psi_{U_3}}(g) = \int_{U_3'(F)\backslash U_3'(\mathbf{A})} \int_{(F\backslash \mathbf{A})^3} \int_{\mathbf{A}^3} \varphi(x_{32}(t_1)x_{54}(t_2)x_{35}(t_3)u'x_{13}(r_1)x_{23}(r_2)x_{46}(r_3)g) \times \psi_{U_3}(u')dr_i dt_i du',$$

where U_3' is the group generated by all the one parameter subgroups X_{ij} contained in U_3, except X_{13}, X_{23}, and X_{46}, and we treat ψ_{U_3} as a character of U_3' by restriction. This is done via arguments completely analogous to those before (9) in Section 2.

Plugging (26) into (25), and making a change of variables in g, we obtain (27)

$$\int_{Z(\mathbf{A})M_3(F)N_3'(F)\backslash G(\mathbf{A})} \left(\int_{(F\backslash \mathbf{A})^3} \int_{U_3'(F)\backslash U_3'(\mathbf{A})} \varphi(x_{32}(t_1)x_{54}(t_2)x_{35}(t_3)u'g)du'dt_i \times \int_{\mathbf{A}} f_{\chi_1'}(w_0w_1^{-1}w_2^{-1}x_{13}(r_1)g)f_{\chi_2'}(w_1^{-1}w_2^{-1}x_{13}(r_1)g)dr_1 \right) \chi_3'(\lambda(g))dg,$$

where $N_3 = X_{23}X_{46}N_3'$. (It is, perhaps, worth noting that $X_{32}X_{35}X_{54}U_3'$ is not a group.)

We conjugate by $w_3 = w[1254]$, which takes X_{32} to X_{13}, X_{35} to X_{14}, X_{54} to X_{46}, and U_3' to the subgroup of U consisting of the product of all X_{ij} except $(i,j) = (1,2); (1,3); (1,4); (1,6);\ (1,8); (4,6); (5,6); (5,7)$. It also takes M_3 to a group containing a copy of GL_2 embedded so that the image of $\left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right)$ is $x_{16}(r)$. Note that $X_{57} = X_{68}$. We expand on $X_{18}X_{68}$. The trivial character contributes zero, the remaining characters are permuted by our GL_2, and the stabilizer of $\psi(u_{57})$ contains X_{16}. We factor this integration. We also note that $w_0w_1^{-1}w_2^{-1}x_{13}(r_1)w_2w_1w_0^{-1} \subset Q$. We get

$$(28)\quad \int_{Z(\mathbf{A})M_4(F)N_4(F)\backslash G(\mathbf{A})} \varphi(g)^{U_4,\psi_{U_4}} f_{\chi_1'}(w_0w_1^{-1}w_2^{-1}w_3^{-1}g) \times \int_{\mathbf{A}} f_{\chi_2'}(w_1^{-1}w_2^{-1}x_{13}(r_1)w_3^{-1}g)dr_1\chi_3'(\lambda(g))dg,$$

where U_4 is the subset of U containing all the X_{ij} except $(i,j) = (1,2)$ and $(5,6)$, $M_4 = \{diag(a,b,b,b,b,c,b,c,c,c,c,a^{-1}bc)\}$, $\psi_{U_4}(u) = \psi(u_{23} + u_{34} + u_{45} + u_{57})$, and N_4 is the product of the X_{ij} for the following (i,j): $i = 1$ or 3, $j \geq 5$, except 8; $i = 2$ or 4, $j = 7, 8$, and $(2,4)$ and $2, 10$.

Finally, we expand φ on X_{12} and X_{56}, and factor the integration over $N_4(\mathbf{A})\backslash U(\mathbf{A})$. Plugging in

$$w_0 w_1^{-1} w_2^{-1} w_3^{-1} = w[43423156],$$

and

$$w_1^{-1} w_2^{-1} w_3^{-1} = w[13643524],$$

as well as $w_3 x_{13}(r_1) w_3^{-1} = x_{21}(r_1)$, yields (21). ∎

5. The Unramified Calculation for GSO_{12}

We now consider the local unramified integral which results from (21). In this section F will denote a non-archimedean local field, π an unramified irreducible representation of $G(F)$, with trivial central character, and $f^R_{\chi'_1}$ and $f^L_{\chi'_2}$ will denote the local analogues of the global functionals defined above. As in Section 2.1, we suppress the "(F)" from the notation. The integral we consider is

$$(29) \qquad \int_{ZU\backslash G} W_\pi(g) f^L_{\chi'_1}(g) f^R_{\chi'_2}(g) \chi'_3(\lambda(g)) dg.$$

The main result of this section is

Proposition: *The integral* (29) *is equal to*

$$(30) \qquad \frac{L(5s_2 - 2, \pi \otimes \chi_2, St) L(4s_1 - \frac{3}{2}, \pi \otimes \chi_1, Spin)}{N(s)}$$

$$N(s) = L(8s_1, \chi_1^2 \omega_\pi) L(8s_1 - 1, \chi_1^2 \omega_\pi) L(8s_1 - 2, \chi_1^2 \omega_\pi)^2 L(16s_1 - 6, \chi_1^4 \omega_\pi^2) \times L(10s_2, \chi_2^2) L(10s_2 - 2, \chi_2^2) L(10s_2 - 4, \chi_2^2).$$

Once again, the denominator matches the product of the normalizing factors of the Eisenstein series exactly.

Proof. As before, we invoke the Iwasawa decompostion and compute $f^R_{\chi'_1}$ and $f^L_{\chi'_2}$. We omit many details, but remark that the addition of characters does not make matters more complicated: since we assume all data is unramified, we may define $u_i, i = 1, 2$ so that $\chi'_i(r) = |r|^{u_i}$. The only new wrinkle is the handling of the variables l_1 and l_2 in the definition of $f^L_{\chi_2'}$, which is as follows: $f^L_{\chi_2'}(t)$ is equal to

$$\int_{F^8} f_{\chi_2'}(w[643524]x_{21}(l_1)x_{23}(l_3)x_{25}(l_4)x_{26}(l_5)x_{29}(l_6)x_{45}(l_7)x_{46}(l_8)t) \psi(r_1l_2+l_3+l_7)dr_i.$$

The integration in l_1 amounts to taking the Fourier transform in this variable. The function $l_1 \mapsto f_{\chi_2'}(w[643524]x_{21}(l_1)h)$ is easily seen to be L^1 and L^2 for s_2 sufficiently large by plugging in the Iwasawa decomposition of $x_{21}(l_1)$ and noting that the smooth function $f_{\chi_2'}$ is bounded on the compact set Kh. The integration in l_2 then gives the value of the original function at 0. Other than this the computation is the same as before. The outcome is

$$f^R_{\chi_1'}(t) = \frac{L(8s_1-3,\chi_1^2\omega_\pi)^4}{L(8s_1,\chi_1^2\omega_\pi)L(8s_1-1,\chi_1^2\omega_\pi)L(8s_1-2,\chi_1^2\omega_\pi)^2} \times \Big(1-\chi_1''\Big(\frac{t_1}{t_2}p\Big)\Big)\Big(1-\chi_1''\Big(\frac{t_3}{t_4}p\Big)\Big)\Big(1-\chi_1''\Big(\frac{t_5}{t_6}p\Big)\Big)\Big(1-\chi_1''\Big(\frac{t_5}{t_7}p\Big)\Big) \chi_1'\circ d_3(w[3423156]tw[6513243])\Big|\frac{t_1^3t_3^2t_5^4}{t_2t_4^2t_6^3t_7^3}\Big|,$$

$$f^L_{\chi_2'}(t) = \frac{L(10s_2-4,\chi_2^2)^2}{L(10s_2,\chi_2^2)L(10s_2-2,\chi_2^2)}\Big|\frac{t_2^4t_4^3}{t_3t_5^2t_6^3t_7}\Big|\chi_2'\circ d_6(w[643524]t) \times\Big(1-\chi_2''\Big(\frac{t_2}{t_3}p\Big)\Big)\Big(1-\chi_2''\Big(\frac{t_4}{t_5}p\Big)\Big).$$

Where p is a uniformizer, and we have introduced the notation $\chi_1''(r)=\chi_1'(r)|r|^{-3}$, and $\chi_2''(r)=\chi_2'(r)|r|^{-4}$.

While our full integrand is Z-invariant, the individual terms that make it up are not. So, to be quite explicit, we replace the integral over $Z\backslash T$ with one over the subgroup of T consisting of elements of the form

$$t = (\tau_1\tau_2\tau_3\tau_4\tau_5\tau_6, \tau_2\tau_3\tau_4\tau_5\tau_6, \tau_3\tau_4\tau_5\tau_6, \tau_4\tau_5\tau_6, \tau_5\tau_6, \tau_6, \tau_5, \dots),$$

which maps isomorphically onto $Z\backslash T$. We let $K_\pi(t) = W_\pi(t)\delta_B(t)^{-1/2}$. Then the above choice of subgroup will be convenient when we to evaluate $K_\pi(t)$.

We also get

$$\begin{aligned}\delta_B(t) &= |\tau_1^{10}\tau_2^{18}\tau_3^{24}\tau_4^{28}\tau_5^{15}\tau_6^{15}|.\\ d_3(w[3423156]tw[6513243]) &= \tau_2\tau_3\tau_4^2\tau_5^2\tau_6^2,\\ d_6(w[643524]tw[425346]) &= \tau_1\tau_3\tau_5^3\tau_6^4,\end{aligned}$$

and $\lambda(t) = \tau_5\tau_6$.

Collecting everything together, we have shown that (29) is equal to

(31)
$$\frac{L(8s_1-3,\chi_1^2\omega_\pi)^4L(10s_2-4,\chi_2^2)^2}{L(8s_1,\chi_1^2\omega_\pi)L(8s_1-1,\chi_1^2\omega_\pi)L(8s_2-2,\chi_1^2\omega_\pi)^2L(10s_2,\chi_2^2)L(10s_2-2,\chi_2^2)}$$
$$\int_{Z\backslash T} K_\pi(t)\prod_{i=1,3,5,6}(1-\chi_1''(p\tau_i))\prod_{i=2,4}(1-\chi_2''(p\tau_i))\chi_1'(\tau_2\tau_3\tau_4^2\tau_5^2\tau_6^2)\chi_2'(\tau_1\tau_3\tau_5^3\tau_6^4)$$
$$\chi_3'(\tau_5\tau_6)|\tau_1^{-4}\tau_2^{-6}\tau_3^{-10}\tau_4^{-12}\tau_5^{-3}\tau_6^{-7}|^{\frac{1}{2}}\,dt.$$

The value of $K_\pi(t)$ is given by the Casselman–Shalika [C-S] formula as follows: for $i=1,\dots,6$, let $|\tau_i| = q^{-n_i}$, and let ϖ_i denote the ith fundamental weight of $Spin_{12}(\mathbf{C})$. Let $(n_1,n_2,n_3,n_4,n_5,n_6)$ denote the irreducible representation of $Spin_{12}(\mathbf{C})$ with highest weight $n_1\varpi_1+\dots+n_6\varpi_6$. Let a be an integer such that $a\equiv n_5+n_6 \mod 2$. Then there is a unique representation of $GSpin_{12}(\mathbf{C})$ such that $Spin_{12}(\mathbf{C})$ acts by $(n_1,n_2,n_3,n_4,n_5,n_6)$ and every scalar λ acts by λ^a. We denote this representation by $(n_1,n_2,n_3,n_4,n_5,n_6;a)$. So $St = (1,0,0,0,0,0;0)$ and $Spin = (0,0,0,0,1,0;1)$. Then for t as above with $|\tau_i| = q^{-n_i}$, the value of $K_\pi(t)$ is equal to the trace of $(n_1,n_2,n_3,n_4,n_5,n_6;n_5+n_6)$, evaluated at the semisimple conjugacy class in $GSpin_{12}(\mathbf{C})$ associated to the representation π. As before, we abuse notation and refer to this evaluation as $(n_1,n_2,n_3,n_4,n_5,n_6;n_5+n_6)$ also.

Let $x=\chi_2(p)q^{-5s_2+2}, y=\chi_1(p)q^{-4s_1+3/2}, w=\omega_\pi(p)$. Observe that

$$(n_1,n_2,n_3,n_4,n_5,n_6;a+2b) = w^b(n_1,n_2,n_3,n_4,n_5,n_6;a). \tag{32}$$

By the Poincaré identity and work of Brion [Br], we have

$$(1-y^4w^2)L(4s_1-3/2,\pi\otimes\chi_1,Spin) = \sum_{\ell_2,\ell_4,\ell_5=0}^{\infty}(0,\ell_2,0,\ell_4,\ell_5,0;2\ell_2+4\ell_4+\ell_5)y^{2\ell_2+4\ell_4+\ell_5}\frac{(1-(y^2w)^{(\ell_5+1)})}{(1-y^2w)},$$

and

$$(1-x^2)L(5s_2-2,\pi\otimes\chi_2,St)=\sum_{\ell_1=0}^{\infty}(\ell_1,0,0,0,0,0;0)x^{\ell_1}.$$

The stated result is now reduced to the following identity:

(33)
$$\sum_{n_i=0}^{\infty}(n_1,n_2,n_3,n_4,n_5,n_6;n_5+n_6)\prod_{i=1,3,5,6}\frac{1-(y^2w)^{(n_i+1)}}{1-y^2w}\prod_{i=2,4}\frac{1-x^{2(n_i+1)}}{1-x^2}$$
$$x^{n_1+n_3+n_6}y^{2n_2+2n_3+4n_4+n_5+n_6}w^{n_2+n_3+2n_4}$$
$$=\sum_{\ell_1,\ell_2,\ell_4,\ell_5=0}^{\infty}(0,\ell_2,0,\ell_4,\ell_5,0;2\ell_2+4\ell_4+\ell_5)(\ell_1,0,0,0,0,0;0)$$
$$x^{\ell_1}y^{2\ell_2+4\ell_4+\ell_5}\frac{1-(y^2w)^{(\ell_5+1)}}{1-y^2w}.$$

If the central character of π is trivial, this reads

(34)
$$\sum_{n_i=0}^{\infty}(n_1,n_2,n_3,n_4,n_5,n_6)\prod_{i=1,3,5,6}\frac{1-y^{2(n_i+1)}}{1-y^2}\prod_{i=2,4}\frac{1-x^{2(n_i+1)}}{1-x^2}$$
$$x^{n_1+n_3+n_6}y^{2n_2+2n_3+4n_4+n_5+n_6}$$
$$=\sum_{\ell_1,\ell_2,\ell_4,\ell_5=0}^{\infty}(0,\ell_2,0,\ell_4,\ell_5,0)(\ell_1,0,0,0,0,0)x^{\ell_1}y^{2\ell_2+4\ell_4+\ell_5}\frac{1-y^{2(\ell_5+1)}}{1-y^2}.$$

We show first that (34) implies (33). To do this, we replace each of the rational functions by a sum, e.g.

$$\frac{1-(y^2w)^{(n_i+1)}}{1-y^2w}=\sum_{k_i=0}^{n_i}y^{2k_i}w^{k_i}.$$

We then use (32) to eliminate w entirely obtaining an identity of power series in x and y alone. Then (33) amounts to a formula for the decomposition of $Sym^a(Spin)\otimes Sym^b(St)$ into irreducibles, and (34) is the same formula for restrictions to $Spin_{12}(\mathbf{C})$. The only information lost is the action of scalars, and this is easily recovered: since scalars act trivially on St and by their first powers on $Spin$, they will act by their ath powers on every constituent of $Sym^a(Spin)\otimes Sym^b(St)$. This is reflected in the power series as the property that when we expand the rational functions and absorb the w's, then every term on both sides has the property that the quantity after the semicolon is equal to the exponent of y.

Next, we prove (34). To do this we make use of work of Black, King, and Wybourne [B-K-W]. Note that the relevant results have also been reformulated in the appendix to [Ga-H], so as to make the meaning of the "modification rules" (see below) more transparent. In their paper, the representation $(n_1, n_2, n_3, n_4, n_5, n_6)$ is denoted by $[\nu]_\pm$ if $n_5 \equiv n_6 \mod 2$, and $[\Delta; \nu]_\pm$ if not, where $n_i = \nu_i - \nu_{i+1}, i = 1$ to 5, $\nu_6 = \lfloor \frac{|n_5 - n_6|}{2} \rfloor$, and the sign $\pm$ is equal to the sign of $n_6 - n_5$. (When $n_5 - n_6$, we technically don't have a sign, but may safely put either. Here, it will be convenient to put "$-$".) Thus we must break (34) into two pieces corresponding to the "tensor" and "spinor" cases. The "tensor" identity is

$$(35) \qquad \sum_{\substack{n_i=0 \\ n_5 \equiv n_6 \mod 2}}^{\infty} (n_1, n_2, n_3, n_4, n_5, n_6) \prod_{i=1,3,5,6} \frac{1 - y^{2(n_i+1)}}{1 - y^2} \times \prod_{i=2,4} \frac{1 - x^{2(n_i+1)}}{1 - x^2} x^{n_1+n_3+n_6} y^{2n_2+2n_3+4n_4+n_5+n_6}$$
$$= \sum_{\ell_1, \ell_2, \ell_4, \ell_5 = 0}^{\infty} (0, \ell_2, 0, \ell_4, 2\ell_5, 0)(\ell_1, 0, 0, 0, 0, 0) x^{\ell_1} y^{2\ell_2+4\ell_4+2\ell_5} \frac{1 - y^{2(2\ell_5+1)}}{1 - y^2},$$

while the spinor has $n_5 \not\equiv n_6 \mod 2$ on the left side and $2\ell_5 + 1$ replacing $2\ell_5$ on the right side.

The relevant formula from [B-K-W] is

$$[\lambda] \times [\mu]_- = \sum_{\eta, \zeta} [\overline{\eta}; (\lambda/\zeta\eta B) \cdot (\mu/\zeta)]_-.$$

In our case, the value of μ is given by

$$(\ell_2 + \ell_4 + \ell_5)^2 (\ell_4 + \ell_5)^2 \ell_5^2,$$

while λ is the partition with one part ℓ_1. So ζ and η must also have one part, and the B just goes away. We get

$$\sum_{i=0}^{\ell_1} \sum_{j=0}^{\ell_1 - i} [\overline{i}; (\ell_1 - i - j) \cdot (\mu/j)]_-.$$

We next note that

$$\mu/j = \sum_a \sigma(\ell, a),$$

where the sum is over triples $a = (a_2, a_4, a_6)$ of nonnegative integers satisfying

$$a_2 \le \ell_2, \qquad a_4 \le \ell_4, \qquad a_6 \le \ell_5, \qquad a_2 + a_4 + a_6 = j,$$

and

$$\sigma(a,\ell) = \ell_2 + \ell_4 + \ell_5, \ell_2 - a_2 + \ell_4 + \ell_5, \ell_4 + \ell_5, \ell_4 - a_4 + \ell_5, \ell_5, \ell_5 - a_6,$$

and hence that

$$(\ell_1 - i - j) \cdot (\mu/j) = \sum_{a,b} \tau(\ell, a, b)$$

where the sum is over 10-tuples $a, b = (a_2, \dots, b_7)$ of nonnegative integers satisfying

$$a_2 + \cdots + b_7 = \ell_1 - i, \qquad b_2 \le a_2, \qquad a_2 + b_3 \le \ell_2,$$
$$b_4 \le a_4, \qquad a_4 + b_5 \le \ell_4, \qquad b_6 \le a_6, \qquad a_6 + b_7 \le \ell_5,$$

and

$$\tau(\ell, a, b) = \ell_2 + \ell_4 + \ell_5 + b_1, \quad \ell_2 - a_2 + b_2 + \ell_4 + \ell_5,$$
$$\ell_4 + \ell_5 + b_3, \quad \ell_4 - a_4 + b_4 + \ell_5, \quad \ell_5 + b_5, \quad \ell_5 - a_6 + b_6, b_7.$$

Thus, the original sum is equal to

$$\sum_{\ell,a,b} \left[\ell_1 - \sum_k a_k - \sum_k b_k ; \tau(\ell, a, b) \right]_{-} x^{\ell_1} y^{2\ell_2 + 4\ell_4 + 2\ell_5} \frac{1 - y^{2(2\ell_5+1)}}{1 - y^2},$$

where the sum is over 14-tuples $(\ell_1, \dots, b_7)$ of nonnegative integers, satisfying the inequalities above, and the additional condition

$$\ell_1 - \sum_k a_k - \sum_k b_k \ge 0.$$

Now, we must apply modification rules. As noted on p. 1581 of [B-K-W], it is necessary to first apply the modification rule for U_6 to obtain a pair of partitions with six or fewer total parts, and then apply the one for SO_{12} to obtain a single partition.

There are four possibilities:

1. If $\ell_1 - \sum_k a_k - \sum_k b_k = b_7 = 0$ then no modification is necessary.
2. If $\ell_1 - \sum_k a_k - \sum_k b_k = b_7 = 1$, then the U_6 modification rule simply deletes these two 1's and introduces a minus sign. (These terms will cancel some of the terms of the first type.) The result does not need to be modified further.

3. If $\ell_1-\sum_k a_k-\sum_k b_k>0$, and $b_7=b_6=\ell_5-a_6=0$, then the U_6 modification rule leaves it alone, and the SO_{12} rule does not. We go into this in more detail below, but the main thing here is that this will produce characters of the form $[\nu]_+$, not $[\nu]_-$. Hence there is no cancellation between these, and the ones coming from the first two.
4. In all other cases, the U_6 modification rule gives 0.

Equation (35) thus splits further into two parts

$$\sum_{\substack{n_i=0\\ n_5\equiv n_6 \mod 2\\ n_5\geq n_6}}^{\infty}(n_1,n_2,n_3,n_4,n_5,n_6)\prod_{i=1,3,5,6}\frac{1-y^{2(n_i+1)}}{1-y^2} \tag{36}$$
$$\times\prod_{i=2,4}\frac{1-x^{2(n_i+1)}}{1-x^2}x^{n_1+n_3+n_6}y^{2n_2+2n_3+4n_4+n_5+n_6}$$
$$=\sum_{\ell,a,b}{}^{(1)}(0,\ell_2,0,\ell_4,2\ell_5,0)(\ell_1,0,0,0,0,0)x^{\ell_1}y^{2\ell_2+4\ell_4+2\ell_5}\frac{1-y^{2(2\ell_5+1)}}{1-y^2}$$
$$+\sum_{\ell,a,b}{}^{(2)}(0,\ell_2,0,\ell_4,2\ell_5,0)(\ell_1,0,0,0,0,0)x^{\ell_1}y^{2\ell_2+4\ell_4+2\ell_5}\frac{1-y^{2(2\ell_5+1)}}{1-y^2},$$

and

$$\sum_{\substack{n_i=0\\ n_5\equiv n_6 \mod 2\\ n_5< n_6}}^{\infty}(n_1,n_2,n_3,n_4,n_5,n_6)\prod_{i=1,3,5,6}\frac{1-y^{2(n_i+1)}}{1-y^2} \tag{37}$$
$$\times\prod_{i=2,4}\frac{1-x^{2(n_i+1)}}{1-x^2}x^{n_1+n_3+n_6}y^{2n_2+2n_3+4n_4+n_5+n_6}$$
$$=\sum_{\ell,a,b}{}^{(3)}(0,\ell_2,0,\ell_4,2\ell_5,0)(\ell_1,0,0,0,0,0)x^{\ell_1}y^{2\ell_2+4\ell_4+2\ell_5}\frac{1-y^{2(2\ell_5+1)}}{1-y^2},$$

where $\sum^{(i)}$ denotes summation with the additional conditions given in case i above. We turn first to (36). In each sum, we make the change of variables

$$a_2'=a_2-b_2,\ a_4'=a_4-b_4,\ a_6'=a_6-b_6,\ \ell_2'=\ell_2-a_2-b_3=\ell_2-a_2'-b_2-b_3,$$
$$\ell_4'=\ell_4-a_4-b_5=\ell_4-a_4'-b_4-b_5,\quad \ell_5'=\ell_5-a_6=\ell_5-a_6'-b_6.$$

The conditions previously imposed are equivalent to the requirement that all of these new variables be nonnegative. We collect the terms corresponding to

ℓ, a, b such that $\tau(\ell, a, b) = \nu$, where ν is given in terms of the n_i as above. Specifically, this amounts to

$$n_1 = b_1 + a_2', \qquad n_2 = b_2 + \ell_2', \qquad n_3 = b_3 + a_4',$$
$$n_4 = b_4 + \ell_4', \qquad n_6 = b_5 + a_6', \qquad \frac{n_5 - n_6}{2} = b_6 + \ell_5'.$$

Now,

$$2\ell_2 + 4\ell_4 + 2\ell_5 = 2\ell_2' + 2a_2' + 2b_2 + 2b_3 + 4\ell_4' + 4a_4' + 4b_4 + 4b_5 + 2\ell_5' + 2a_6' + 2b_6$$
$$= 2n_2 + 2n_3 + 4n_4 + n_5 + n_6 + 2a_2' + 2a_4' + 2b_5.$$

In the first sum,

$$\ell_1 = a_2' + a_4' + a_6' + b_1 + 2b_2 + b_3 + 2b_4 + b_5 + 2b_6 = n_1 + n_3 + n_6 + 2b_2 + 2b_4 + 2b_6,$$

while in the second sum, it is equal to this same quantity plus 2. Also, the first sum is over all a', b, ℓ' satisfying the equalities above, while the second has the additional condition $\ell_5' > 0$. We may express each sum as a sum over $a_2', a_4', a_6', b_2, b_4, b_6$. The shift $b_6 \to b_6 + 1$ makes the exponents agree. Taking the difference, leaves only the terms corresponding to $b_6 = 0$. The summation in a_2', a_4', b_2, and b_4 is straightforward, and yields

$$\frac{1 - x^{2(n_2+1)}}{1 - x^2} \frac{1 - x^{2(n_4+1)}}{1 - x^2} \frac{1 - y^{2(n_1+1)}}{1 - y^2} \frac{1 - y^{2(n_3+1)}}{1 - y^2}.$$

The summation in a_6' is

$$\sum_{a_6'=0}^{n_6} y^{2(n_6 - a_6')} \frac{1 - y^{2(n_5 - n_6 + 2a_6') + 1}}{1 - y^2}$$
$$= (1 - y^2)^{-1} \Big(\sum_{a_6'=0}^{n_6} y^{2(n_6 - a_6')} - y^{2(n_5+1)} \sum_{a_6'=0}^{n_6} y^{2a_6'} \Big)$$
$$= \frac{1 - y^{2(n_5+1)}}{1 - y^2} \frac{1 - y^{2(n_6+1)}}{1 - y^2}.$$

This completes the proof of (36).

We now turn to (37). We first compute the contribution to (37) corresponding to a fixed pair $\{\bar{s}; \tau\}$. Let $m_i = \tau_i - \tau_{i+1}$, for $1 \le i \le 4$. And let $a_i', \ell_i', i = 2, 4, 6$ be defined as before The sum over triples ℓ, a, b, such that $\ell_5 - a_6 = b_6 = b_7 = 0$

and $\{\overline{\ell_1 - \sum_k a_k - \sum_k b_k}; \tau(\ell, a, b)\} = \{\bar{s}, \tau\}$ is equal to the sum over ℓ, a, b subject to the following conditions

$$a_2' + b_1 = m_1, \quad \ell_2' + b_2 = m_2, \quad a_4' + b_3 = m_3, \quad \ell_4' + b_4 = m_4, \quad \ell_5' + b_5 = \tau_5,$$
$$\ell_1 = s + \ell_5 + a_2' + a_4' + b_1 + 2b_2 + b_3 + 2b_4 + b_5 = s + m_1 + m_3 + \tau_5 + 2b_2 + 2b_4.$$

Furthermore,

$$2\ell_2 + 4\ell_4 + \ell_5 = 2m_2 + 2m_3 + 4m_4 + 4\tau_5 + 2a_2' + 2a_4' - 2\ell_5.$$

The sums on b_2, b_4, a_2' and a_4' yield

$$\frac{1-x^{2(m_2+1)}}{1-x^2}\frac{1-x^{2(m_4+1)}}{1-x^2}\frac{1-y^{2(m_1+1)}}{1-y^2}\frac{1-y^{2(m_3+1)}}{1-y^2},$$

and

$$\sum_{\ell_5=0}^{\tau_5} y^{-2\ell_5}\frac{1-y^{2(2\ell_5+1)}}{1-y^2} = (1-y^2)^{-1}\left(\sum_{\ell_5=0}^{\tau_5} y^{-2\ell_5} + y^2\sum_{\ell_5=0}^{\tau_5} y^{2\ell_5}\right)$$
$$= y^{-2\tau_5}\left(\frac{1-y^{2(\tau_5+1)}}{1-y^2}\right)^2,$$

so overall we get

$$x^{m_1+m_3+\tau_5+s}y^{2m_2+2m_3+4m_4+2\tau_5}\frac{1-x^{2(m_2+1)}}{1-x^2}\frac{1-x^{2(m_4+1)}}{1-x^2} \times \frac{1-y^{2(m_1+1)}}{1-y^2}\frac{1-y^{2(m_3+1)}}{1-y^2}\left(\frac{1-y^{2(\tau_5+1)}}{1-y^2}\right)^2. \tag{38}$$

There are six pairs $\{\overline{s}; \tau\}$ such that under the SO_{12} modification rule $[\overline{s}; \tau]_- = \pm[\nu]_+$, where ν is associated to n_i as above. They are:

$$\{\overline{\nu_6}; \nu_1, \nu_2, \nu_3, \nu_4, \nu_5\}$$
$$\{\overline{\nu_5 + 1}; \nu_1, \nu_2, \nu_3, \nu_4, \nu_6 - 1\}$$
$$\{\overline{\nu_4 + 2}; \nu_1, \nu_2, \nu_3, \nu_5 - 1, \nu_6 - 1\}$$
$$\{\overline{\nu_3 + 3}; \nu_1, \nu_2, \nu_4 - 1, \nu_5 - 1, \nu_6 - 1\}$$
$$\{\overline{\nu_2 + 4}; \nu_1, \nu_3 - 1, \nu_4 - 1, \nu_5 - 1, \nu_6 - 1\}$$
$$\{\overline{\nu_1 + 5}; \nu_2 - 1, \nu_3 - 1, \nu_4 - 1, \nu_5 - 1, \nu_6 - 1\}$$

The signs $\pm$ alternate, starting with plus. The corresponding values of m_i, τ_5, $m_1 + m_3 + s$, and $2m_2 + 2m_3 + 4m_4 + 2\tau_5$ are as follows:

m	τ_5	$m_1 + m_3 + \tau_5 + s$	$2m_2 + 2m_3 + 4m_4 + 2\tau_5$
n_1, n_2, n_3, n_4	$\frac{n_6+n_5}{2}$	$n_1 + n_3 + n_6$	$2n_2 + 2n_3 + 4n_4 + n_5 + n_6$
$n_1, n_2, n_3, n_4 + n_5 + 1$	$\frac{n_6-n_5}{2} - 1$	$n_1 + n_3 + n_6$	$2n_2 + 2n_3 + 4n_4 + 3n_5 + n_6 + 2$
$n_1, n_2, n_3 + n_4 + 1, n_5$	$\frac{n_6-n_5}{2} - 1$	$n_1 + n_3 + 2n_4 + n_6 + 2$	$2n_2 + 2n_3 + 2n_4 + 3n_5 + n_6$
$n_1, n_2 + n_3 + 1, n_4, n_5$	$\frac{n_6-n_5}{2} - 1$	$n_1 + n_3 + 2n_4 + n_6 + 2$	$2n_2 + 2n_3 + 2n_4 + 3n_5 + n_6$
$n_1 + n_2 + 1, n_3, n_4, n_5$	$\frac{n_6-n_5}{2} - 1$	$n_1 + 2n_2 + n_3 + 2n_4 + n_6 + 4$	$2n_3 + 2n_4 + 3n_5 + n_6 - 2$
n_2, n_3, n_4, n_5	$\frac{n_6-n_5}{2} - 1$	$n_1 + 2n_2 + n_3 + 2n_4 + n_6 + 4$	$2n_3 + 2n_4 + 3n_5 + n_6 - 2$

Now we plug these six sets of values into (38), and introduce the notation $X_i = x^{n_i}, Y_i = y^{n_i}$. The identity (37) is reduced to the following equality of polynomials:

$$
\begin{aligned}
&(1 - X_2^2x^2)(1 - X_4^2x^2)(1 - Y_1^2y^2)(1 - Y_3^2y^2)(1 - Y_5Y_6y^2)^2Y_2^2Y_4^2 \\
&- (1 - X_2^2x^2)(1 - X_4^2X_5^2x^4)(1 - Y_1^2y^2)(1 - Y_3^2y^2)(1 - Y_5^{-1}Y_6)^2Y_2^2Y_4^2Y_5^2y^2 \\
&+ (1 - X_2^2x^2)(1 - X_5^2x^2)(1 - Y_1^2y^2)(1 - Y_3^2Y_4^2y^4)(1 - Y_5^{-1}Y_6)^2X_4^2x^2Y_2^2Y_5^2 \\
&- (1 - X_2^2X_3^2x^4)(1 - X_5^2x^2)(1 - Y_1^2y^2)(1 - Y_4^2y^2)(1 - Y_5^{-1}Y_6)^2X_4^2x^2Y_2^2Y_5^2 \\
&+ (1 - X_3^2x^2)(1 - X_5^2x^2)(1 - Y_1^2Y_2^2y^4)(1 - Y_4^2y^2)(1 - Y_5^{-1}Y_6)^2X_2^2X_4^2x^4Y_5^2y^{-2} \\
&- (1 - X_3^2x^2)(1 - X_5^2x^2)(1 - Y_2^2y^2)(1 - Y_4^2y^2)(1 - Y_5^{-1}Y_6)^2X_2^2X_4^2x^4Y_5^2y^{-2} \\
&= (1 - X_2^2x^2)(1 - X_4^2x^2)(1 - Y_1^2y^2)(1 - Y_3^2y^2)(1 - Y_5^2y^2)(1 - Y_6^2y^2)Y_2^2Y_4^2.
\end{aligned}
$$

This is not hard to check: if one starts at the bottom and works one's way up, then at each stage, the sum only differs from the next term to be added in a few places. For example, the difference of the last two terms consists of a large part that is the same in both:

$$(1 - X_3^2x^2)(1 - X_5^2x^2)(1 - Y_4^2y^2)(1 - Y_5^{-1}Y_6)^2X_2^2X_4^2x^4Y_5^2y^{-2}$$

times a simple difference:

$$((1 - Y_1^2Y_2^2y^4) - (1 - Y_2^2y^2)) = Y_2^2y^2(1 - Y_1^2y^2).$$

We get

$$(1 - X_3^2x^2)(1 - X_5^2x^2)(1 - Y_1^2y^2)(1 - Y_4^2y^2)(1 - Y_5^{-1}Y_6)^2X_2^2X_4^2x^4Y_2^2Y_5^2,$$

which now has many terms in common with the third-to-last term, and so on. The "spinor" case is handled similarly. (Note that the SO_{12} modification rule is different in the "spinor case.") ∎

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